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1971 J. Phys. A: Gen. Phys. 4 517

(http://iopscience.iop.org/0022-3689/4/4/015)

J. Phys. A: Gen. Phys., 1971, Vol. 4. Printed in Great Britain

Statistical accuracy in the digital autocorrelation of photon counting fluctuations

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MS. received 29th December 1970

Abstract. We investigate the statistical errors which arise in digital autocorrelation of photon counting fluctuations due to the finite duration of experiments. An expression for the accuracy with which individual points of the autocorrelation function of intensities can be measured is derived analytically and evaluated for Gaussian–Lorentzian light. The appropriate weighting for a 'least-squares' fitting procedure is determined and the error in spectral linewidth, obtained by this procedure from experiment, is calculated.

1. Introduction

The methods of intensity fluctuation spectroscopy are now being applied to a wide variety of physical problems including measurements of critical phenomena, transport, and turbulence (see, for example, Dubin and Benedek 1969, Cummins and Swinney 1970, Pike 1969). In this paper we set out to find the answer to a conceptually very simple problem which arises when we attempt to employ these new methods, in particular that of digital autocorrelation of photon counting fluctuations (Jakeman 1970, Foord *et al.* 1970), to measure spectral linewidths. This problem may be stated as follows: for a given spectral linewidth and a given total light flux, what is the accuracy of the value obtained for the linewidth in a single experiment, as a function of the total duration of the experiment and the sample time used? The answer to this question is clearly of vital interest for practical applications; it would allow one to make optimum choices of expensive laser power and store capacity in particular circumstances, as well as informing the experimenter with given equipment of the time necessary to achieve a predetermined accuracy of measurement.

Although conceptually simple the solution is quite lengthy and difficult. The fluctuations being sampled arise from both the statistical nature of the photodetection process and the intensity fluctuations due to the statistics and spectrum of the light field. We have not achieved a complete solution but give an approximation to it which we believe is sufficiently accurate to provide the guidance required in the planning of experiments. Preliminary results were presented in an earlier publication (Jakeman *et al.* 1970b). In the course of the calculation we find the complete solution to the simpler problem of the accuracies of the measured values of the true intensity autocorrelation function (as distinct from any clipped form: Jakeman and Pike 1969) and their dependence on the parameters mentioned above, and on the delay time, for the case of spatially coherent detection. We also find a suitable weighting to be given in using these values to determine the spectral linewidth and investigate the effects of clipping and finite detector area on the results.

To clarify the nature of the calculations to be carried out and to introduce the notation it is appropriate at this point to outline briefly the sampling scheme used in digital autocorrelation of photon counting fluctuations. Referring to figure 1, samples of the autocorrelation function are constructed at intervals T_p, each sample consisting of M channels containing the product $n(0; T)\, n(rT; T)$ where the integer r runs from

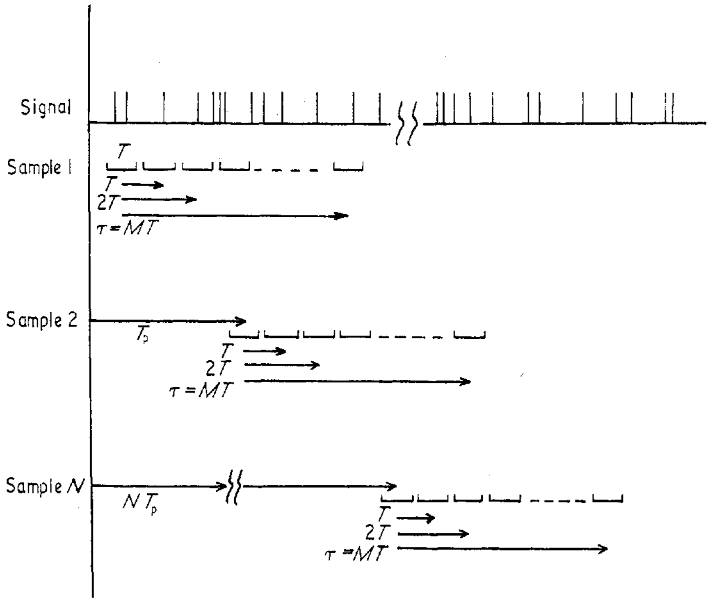

Figure 1. Sampling scheme used by Foord *et al.* (1970).

1 to M, and $n\ (t;\ T)$ is the number of photons counted in the interval T centred at time t. The integrated intensity $E(t;\ T)$ is defined in terms of the instantaneous intensity,

$$I(t) = \mathscr{E}^{+}(t)\mathscr{E}^{-}(t) \tag{1}$$

where $\mathscr{E}^{+}$ and $\mathscr{E}^{-}$ are the positive and negative frequency components of the field, assumed to have only one polarized component, by

$$E(t;\ T) = \int_{t-T/2}^{t+T/2} I(t')\, \mathrm{d}t'. \tag{2}$$

It is not difficult to establish the following relationship between the autocorrelation functions, respectively, of photon counting fluctuations and integrated intensity fluctuations:

$$G^{(2)}(\tau;\ T) = \langle n(0;\ T)n(\tau;\ T)\rangle = \alpha^2 \langle E(0;\ T)E(\tau;\ T)\rangle \qquad \tau \geqslant T \tag{3}$$

$$G^{(2)}(0;\ T) = \langle n(0;\ T)(n(0;\ T)-1)\rangle = \alpha^2 \langle E^2(0;\ T)\rangle \qquad \tau = 0$$

where α is the quantum efficiency of the detector and we have assumed stationarity of the scattering processes. For simplicity we consider for the moment detection at a single space point only; the effects of spatial coherence will be discussed in § 4. The sample time T is the resolution time of the instrument if, as we shall assume, dead-time effects are negligible. We shall assume that τ is a multiple of T_p when $T_p < \tau$. This ensures that the information contained in one sample does not form part of the information contained in another sample. This general scheme of sampling has been considered in the first part of our calculations but in the instruments in use in this laboratory each sample is constructed within the time T and sampling is carried out with a period $T_p = T$ to minimize information wastage.

In the next section we solve the problem of the accuracy of the values of the autocorrelation function measured using the above sampling scheme, firstly in general terms and then for the case of Gaussian–Lorentzian light. We discuss a number of limiting cases which can be treated analytically, whilst the more general results are presented graphically in figure 2. Drift of the mean photo-electron count

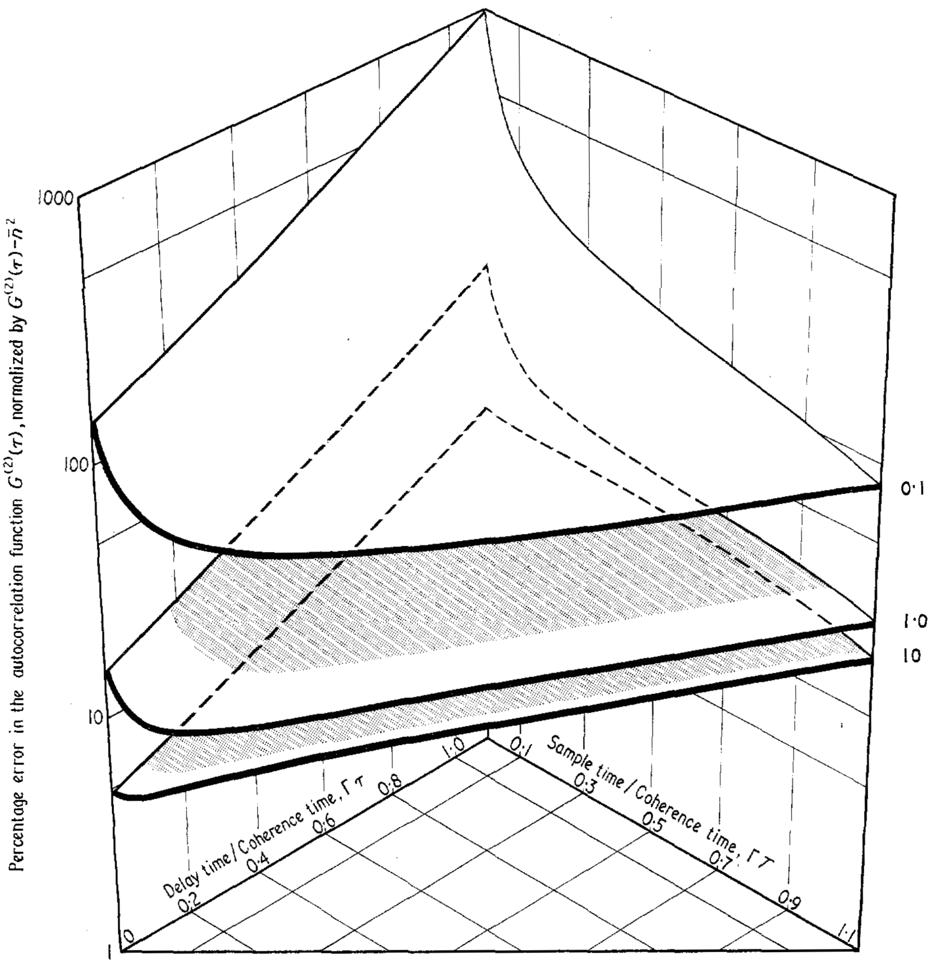

Figure 2. Normalized variance of $G^{(2)}(\tau)$ as a function of delay time and sample time for the values of the photon flux shown per coherence time. The experimental time is given by $N\Gamma T = 10^4$, the ordinate scales as $N^{-1/2}$.

rate during the course of a series of experiments necessitates normalization of each complete measurement. This leads to a biased estimate of the normalized autocorrelation function and its effects will also be discussed in § 2. In § 3 we tackle the problems of the errors in the measured values of linewidth, presenting the main results of the paper in graphical form in figures 3 and 4. Results are given for 20, 100, and effectively an infinite number of channels. We explain the approximations used to obtain these results and give useful analytic forms for certain limiting cases. The effects of clipping and finite detector area are investigated in § 4 and the final section is devoted to discussion of the graphs and formulae presented.

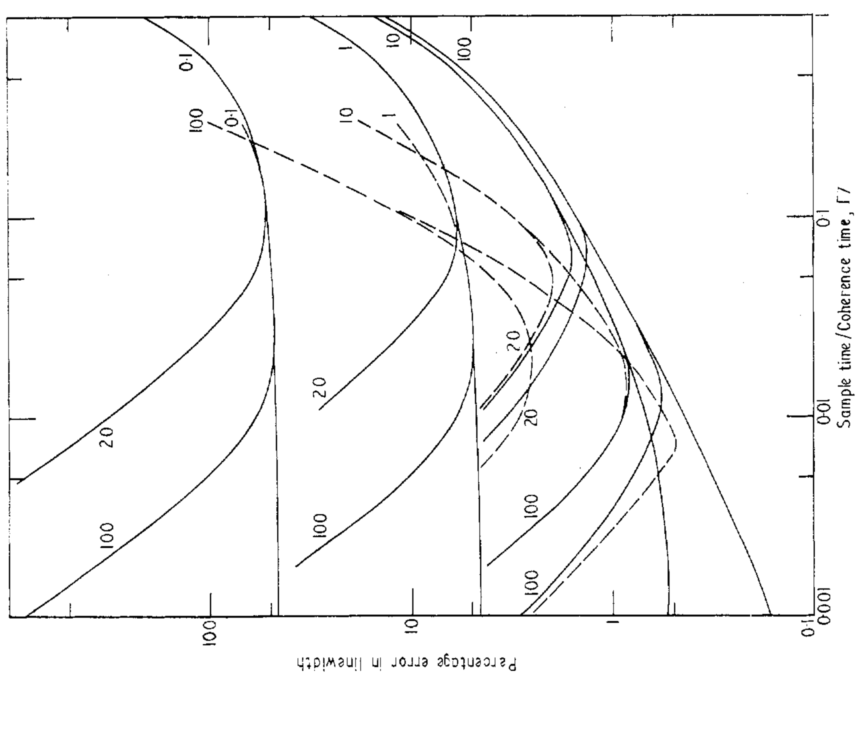

Figure 3. The percentage error in linewidth as a function of sample time for the photon flux and number of channels shown on the right hand and left hand side of the graph respectively: ——— unclipped; - - - - - clipped at zero. When $r = 0{\cdot}1$ and 1 the clipped and unclipped results coincide for $\Gamma T \leqslant 0{\cdot}1$. $N\Gamma T = 10^4$.

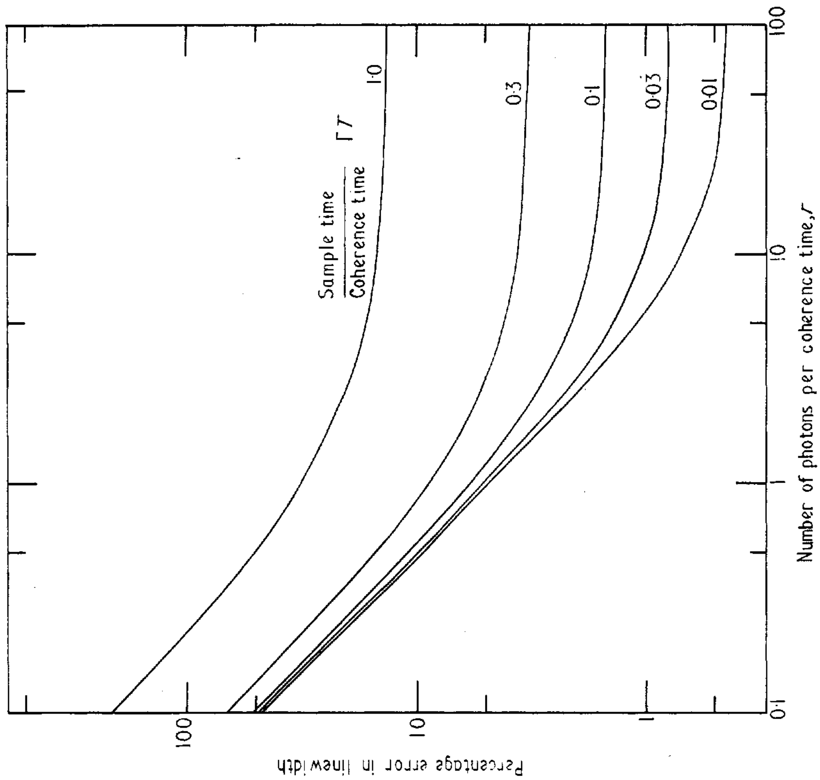

Figure 4. The percentage error in linewidth as a function of photon flux for the values of ΓT shown and an infinite number of channels in the unclipped case. $N\Gamma T = 10^4$

2. Statistics of the intensity autocorrelation estimator

An unbiased estimator for the un-normalized second-order correlation function defined by equation (3) is

$$\hat{G}^{(2)}(\tau) = \frac{1}{N}\sum_{r=1}^{N} n(t_r+\tau)n(t_r). \tag{4}$$

Here N is the total number of products constructed with the delay time τ, and for simplicity we have dropped the sample time T from our notation. It is our purpose in this section to calculate the errors in the estimator $\hat{G}^{(2)}(\tau)$ (and in the measurement of the normalized second-order correlation function $g^{(2)}(\tau)$ which is defined later) due to the statistical nature of light and the photodetection process in an experiment of finite duration. We assume that we are dealing with a stationary process and use the sampling scheme outlined in figure 1. Making use of the known results for this case the variance of $\hat{G}^{(2)}(\tau)$ may be written as follows (Davenport and Root 1958)

$$\operatorname{Var}\hat{G}^{(2)}(\tau) = \frac{1}{N}\operatorname{Var}(n(\tau)n(0)) + \frac{2}{N}\sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)\{R(kT_\mathrm{p}) - |G^{(2)}(\tau)|^2\} \tag{5}$$

where

$$R(kT_\mathrm{p}) = \langle n(kT_\mathrm{p}+\tau)n(kT_\mathrm{p})n(\tau)n(0)\rangle. \tag{6}$$

As we have already pointed out, in an actual experiment the mean count rate may drift with time for reasons unconnected with the fundamental statistical properties of the signal, and it is usual to try to allow for this by determining the quantity

$$\hat{g}^{(2)}(\tau) = \frac{(1/N)\Sigma_{r=1}^{N} n(t_r+\tau)n(t_r)}{\{(1/N)\Sigma_{r=1}^{N} n(t_r)\}^2}. \tag{7}$$

This is a 'biased' estimator for the normalized second-order correlation function

$$g^{(2)}(\tau) = \frac{1}{\bar{n}^2}\langle n(\tau)n(0)\rangle \tag{8}$$

where $\bar{n}$ is the mean number of photo-electrons detected per sample. (By 'biased' we mean $\langle\hat{g}^{(2)}(\tau)\rangle \neq g^{(2)}(\tau)$). The degree of bias may be investigated by expanding the denominator of the right hand side of equation (7) about the mean before averaging. For large values of N this gives

$$\langle\hat{g}^{(2)}(\tau)\rangle = g^{(2)}(\tau) + \frac{2}{\bar{n}^3N^2}\sum_{r=1}^{N}\sum_{s=1}^{N}\langle n(t_r+\tau)n(t_r)(\bar{n}-n(t_s))\rangle + \text{higher-order terms.} \tag{9}$$

It can be shown that the first-order correction term to $g^{(2)}(\tau)$ in equation (9) is of order $1/N$, and $\hat{g}^{(2)}(\tau)$ as defined by equation (7) is an unbiased estimator to this order.

By a similar expansion to that used to derive (9) it may be shown that

$$\mathrm{Var}\left(\frac{\hat{G}^{(2)}(\tau)}{\hat{n}^2}\right) \simeq \frac{1}{\bar{n}^4}\mathrm{Var}\,\hat{G}^{(2)}(\tau) + \frac{4(g^{(2)}(\tau))^2}{\bar{n}^2}\mathrm{Var}\,\hat{n} + 4(g^{(2)}(\tau))^2 - \frac{4g^{(2)}(\tau)}{\bar{n}^3}\langle \hat{G}^{(2)}(\tau)\hat{n}\rangle \tag{10}$$

where $\hat{n}$ is the estimator for the mean count rate.

This expression is valid provided that N is so large that the condition

$$\left|\frac{\hat{n}-\bar{n}}{\bar{n}}\right| \ll 1$$

is satisfied (see for example Farley 1969).

We now calculate Var $(\hat{G}^{(2)}(\tau))$ for Gaussian–Lorentzian light using equations (5) and (6), turning our attention first of all to the evaluation of the $R(kT_\mathrm{p})$ term. Throughout the following calculations we shall assume for simplicity that the quantum efficiency of the detector is unity. This does not make the results less general provided they are expressed in terms of photon numbers rather than intensities. In order to include the possibility that $kT_\mathrm{p} = \tau$ for some value of k, say $k = m$, $R(kT_\mathrm{p})$ may be written using the relation between the joint photon counting and intensity fluctuation distributions for Poisson detection (for example Jakeman, 1970, equation (16)), in the form

$$R(kT_\mathrm{p}) = \langle E(kT_\mathrm{p}+\tau)E(kT_\mathrm{p})E(\tau)E(0)\rangle + \delta_{\mathrm{km}}\langle E(2\tau)E(\tau)E(0)\rangle. \tag{11}$$

Using (2) the right hand side becomes

$$\int_{kT_\mathrm{p}+\tau-\frac{1}{2}T}^{kT_\mathrm{p}+\tau+\frac{1}{2}T} \mathrm{d}t_1 \int_{kT_\mathrm{p}-\frac{1}{2}T}^{kT_\mathrm{p}+\frac{1}{2}T} \mathrm{d}t_2 \int_{\tau-\frac{1}{2}T}^{\tau+\frac{1}{2}T} \mathrm{d}t_3 \int_{-\frac{1}{2}T}^{\frac{1}{2}T} \mathrm{d}t_4 \langle I(t_1)I(t_2)I(t_3)I(t_4)\rangle + \delta_{\mathrm{km}} \int_{2\tau-\frac{1}{2}T}^{2\tau+\frac{1}{2}T} \mathrm{d}t_1 \int_{\tau-\frac{1}{2}T}^{\tau+\frac{1}{2}T} \mathrm{d}t_2 \int_{-\frac{1}{2}T}^{\frac{1}{2}T} \mathrm{d}t_3 \langle I(t_1)I(t_2)I(t_3)\rangle. \tag{12}$$

If $kT_\mathrm{p} \neq \tau$ for any value of k, the last term does not contribute. The quantities $\langle I(t_1)I(t_2)I(t_3)I(t_4)\rangle$ and $\langle I(t_1)I(t_2)I(t_3)\rangle$ may be evaluated by using the factorization properties of Gaussian light (Glauber 1965 p. 150) to express them in terms of the normalized first order or field autocorrelation function

$$g^{(1)}(\tau) = \frac{\langle \mathscr{E}^+(\tau)\mathscr{E}^-(0)\rangle}{\langle I\rangle}.$$

For a Lorentzian spectrum of half width at half height Γ and centre frequency ω_0 this takes the form

$$g^{(1)}(\tau) = \exp(-\Gamma|\tau| + \mathrm{i}\omega_0\tau).$$

There is no difficulty in performing the integration in (12) and the sums in (5), but because of the length of some of the expressions obtained, we present the results in the appendix, where we also give explicit expressions for $\langle I(t_1)I(t_2)I(t_3)I(t_4)\rangle$ and $\langle I(t_1)I(t_2)I(t_3)\rangle$.

The first term in equation (5) may be obtained using a somewhat different approach. By definition.

$$\mathrm{Var}(n(\tau)n(0)) = \langle n^2(\tau)n^2(0)\rangle - \langle n(\tau)n(0)\rangle^2. \tag{13}$$

This can be written in terms of the moments of the integrated intensity distribution following the procedure used to obtain equation (11):

$$\begin{aligned}\mathrm{Var}(n(\tau)n(0)) = {} & \langle E^2(\tau)E^2(0)\rangle + \langle E^2(\tau)E(0)\rangle \\ & + \langle E(\tau)E^2(0)\rangle + \langle E(\tau)E(0)\rangle \\ & - \langle E(\tau)E(0)\rangle^2.\end{aligned} \tag{14}$$

To evaluate (14) we need the second-order generating function

$$Q(S, S') = \langle \exp -(ES+E'S')\rangle$$

which has been given for finite sample time T by Jakeman (1970). In terms of this quantity we have

$$\begin{aligned}\mathrm{Var}(n(\tau)n(0)) = {} & \left[\frac{\mathrm{d}^4Q(S, S')}{\mathrm{d}S^2\,\mathrm{d}S'^2} - 2\frac{\mathrm{d}^3Q(S, S')}{\mathrm{d}S^2\,\mathrm{d}S'}\right]_{S=S'=0} \\ & + g^{(2)}(\tau)(1-g^{(2)}(\tau))\end{aligned} \tag{15}$$

which leads, for a Lorentzian spectrum, to the result

$$\begin{aligned}\frac{\mathrm{Var}(n(\tau)n(0))}{\bar{n}^4} = {} & \left(1+\frac{1}{\gamma}-\frac{\mathrm{e}^{-\gamma}\,\mathrm{sh}\,\gamma}{\gamma^2}+\frac{1}{\bar{n}}\right)^2 + \frac{\mathrm{e}^{-2\Gamma\tau}}{\gamma^2} \\ & \times\left[\mathrm{sh}\,\gamma\left\{4+\frac{2}{\gamma}+\frac{1}{\bar{n}}\right\}-2\,\mathrm{ch}\,\gamma\right]^2 + \left(\frac{\mathrm{e}^{-2\Gamma\tau}\,\mathrm{sh}^2\gamma}{\gamma^2}-1\right) \\ & \times\left(3\mathrm{e}^{-2\Gamma\tau}\frac{\mathrm{sh}^2\gamma}{\gamma^2}+1\right).\end{aligned} \tag{16}$$

Here $\gamma = \Gamma T$.

Equation (16), in conjunction with equation (A3), gives us finally the variance of $\hat{G}^{(2)}(\tau)$. In figure 2 we give a three-dimensional plot of the standard deviation, $(\mathrm{Var}\,\hat{G}^{(2)}(\tau))^{1/2}$ divided by $(\hat{G}^{(2)}(\tau)-\bar{n}^2)$, as a function of $\Gamma\tau$ and γ for various values of the photon flux taking $T = T_{\mathrm{p}}$. We have taken $N\gamma = 10^4$ throughout. This corresponds to a fixed experimental time of 10^4 coherence times.

To obtain $\mathrm{Var}\,\hat{g}^{(2)}(\tau)$ it is necessary to evaluate the additional terms given in equation (10), which we denote by Δ:

$$\Delta(\tau) = 4g^{(2)}(\tau)\left\{g^{(2)}(\tau)\left(\frac{\mathrm{Var}\,\hat{n}}{\bar{n}^2}+1\right)-\frac{\langle\hat{G}^{(2)}(\tau)\hat{n}\rangle}{\bar{n}^3}\right\}. \tag{17}$$

Now

$$\langle\hat{G}^{(2)}(\tau)\hat{n}\rangle = \frac{1}{N^2}\sum_r\sum_s\langle n(t_r+\tau)n(t_r)n(t_s)\rangle \tag{18}$$

and

$$\mathrm{Var}\,\hat{n} = \frac{1}{N^2}\sum_r\sum_s\langle n(t_r)n(t_s)\rangle - \bar{n}^2. \tag{19}$$

These quantities may be evaluated using the methods of § 2.1, but the calculations are not given here. Instead we quote the expression for Δ to order $1/N$, and refer the reader to the Appendix for the exact results for $\langle \hat{G}^{(2)}(\tau)\hat{n}\rangle$ and Var $\hat{n}$. Assuming that $N\gamma \gg 1$ and $m \ll N$, Δ is given by

$$\Delta(mT) = \frac{4g^{(2)}(mT)}{N}\left\{-g^{(2)}(mT)\left(\frac{1}{\gamma}+\frac{1}{\bar{n}}\right)-2\frac{\mathrm{sh}^2\gamma}{\gamma^2}\mathrm{e}^{-x}\left(m-1+\frac{1}{\gamma}-\frac{2}{\mathrm{e}^{2\gamma}-1}\right)\right\} \tag{20}$$

where $x = 2\Gamma\tau$, $m = x/2\gamma$.

To be consistent with our discussion of the bias of $\hat{g}^{(2)}(\tau)$, only those terms in (A3) which are of order $1/N$ are retained during the course of its evaluation. To make contact with experiment we also assume that $T_p = T$. We then find

$$\begin{aligned}\frac{N}{\bar{n}^4}\operatorname{Var}\hat{G}^{(2)}(\tau) = {} & \frac{1}{\bar{n}^4}\operatorname{Var}(n(\tau)n(0))+\left(\frac{1-\mathrm{e}^{-2\gamma}}{2\gamma^2}\right)\\ & \times\left\{2+5\mathrm{e}^{-x}-\mathrm{e}^{2\gamma-x}+\frac{\mathrm{sh}^2\gamma}{\gamma^2}\frac{1}{\mathrm{e}^{2\gamma}+1}(1-3\mathrm{e}^{-2x}-6\mathrm{e}^{+2\gamma-2x})\right\}\\ & +2\left(\frac{1}{\gamma}+\frac{1}{\bar{n}}\right)\left\{1+\frac{\mathrm{sh}^2\gamma}{\gamma^2}\mathrm{e}^{-x}(4+3\mathrm{e}^{-x})\right\}+\frac{2\mathrm{sh}^2\gamma}{\gamma^2}\mathrm{e}^{-x}\\ & \times\left\{(m-1)\left(4+3\mathrm{e}^{-x}\frac{\mathrm{sh}^2\gamma}{\gamma^2}\right)-\frac{2}{\bar{n}}\right\}\end{aligned} \tag{21}$$

where Var $(n(\tau)\,n(0))$ is given by equation (16).

It is useful to consider the form Var $(\hat{g}^{(2)}(\tau))$ takes in the following limits:

(i) $\gamma \ll 1$. This is a common experimental situation. We can expand the exponentials in (20) and (21) to obtain the expression

$$\operatorname{Var}[\hat{g}^{(2)}(\tau)] \doteqdot \frac{1}{N\gamma}\left[\tfrac{1}{2}\{1+8\,\mathrm{e}^{-x}-\mathrm{e}^{-2x}(5+2x)\}+\frac{2}{r}(1+\mathrm{e}^{-x})^2+\frac{1+\mathrm{e}^{-x}}{r^2\gamma}\right]. \tag{22}$$

We have written $\bar{n} = r\gamma$ so that r is the number of photons detected in a coherence time. In a given experiment r is determined by the scattering system, the power of the laser, and the direction of observation. To emphasize that we are interested in the experimental situation where the experiment time (that is, $N\gamma$) is fixed and not the total number of samples (N) we have taken out a factor $(N\gamma)^{-1}$.

(ii) $\gamma \ll 1$, $r \ll 1$. Equation (22) becomes very simple if we assume that $\bar{n}$ is so small that the final term dominates. This is the situation in which the shot noise effect of the photon-detection process is the most important. Then

$$\operatorname{Var}\hat{g}^{(2)}(\tau) = \frac{1+\mathrm{e}^{-x}}{(N\gamma)\gamma r^2}. \tag{23}$$

It is interesting to note that one can write this expression in a particularly simple form in terms of $G^{(2)}(\tau)$ as

$$\operatorname{Var}\hat{G}^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{N}. \tag{24}$$

Equation (24) holds independently of the spectrum, because in the region where the 'shot noise' dominates, it is the $\langle E(\tau)E(0)\rangle$ term of the $\mathrm{Var}(n(\tau)n(0))$ contribution (14), to $\mathrm{Var}\ G^{(2)}(\tau)$ (5), which dominates. The bias of $\hat{g}^{(2)}(\tau)$ does not affect its variance in this limit (that is, $\Delta \sim 0$) and

$$\mathrm{Var}\,\hat{g}^{(2)}(\tau) \simeq \frac{1}{\bar{n}^4}\mathrm{Var}\ \hat{G}^{(2)}(\tau).$$

(iii) $\gamma \ll 1$, $r\gamma \gtrsim 1$. In these circumstances, it is the first term of (22) that dominates and

$$\mathrm{Var}\,\hat{g}^{(2)}(\tau) \simeq \frac{1}{2N\gamma}\{1+8\mathrm{e}^{-x}-\mathrm{e}^{-2x}(5+2x)\}. \tag{25}$$

In this limit the strength of the signal is so great that the 'shot-noise' of the photo-detection process is negligible, and it is the statistical nature of the light which determines the variance of $\hat{g}^{(2)}(\tau)$.

(iv) $\gamma \gg 1$. To determine the form of the variance in this limit we must go back to (A3), (20) and (16). We find

$$\mathrm{Var}\ \hat{G}^{(2)}(\tau) \sim \frac{4}{(N\gamma)}\left(1+\frac{1}{r}\right)$$

$$\mathrm{Var}\,\hat{g}^{(2)}(\tau) \sim \frac{1}{(N\gamma)\gamma}\left(1+\frac{1}{r}\right)^2. \tag{26}$$

Thus, although the variance of the estimator for the un-normalized autocorrelation function tends to a constant in the large γ limit, the variance of the biased estimator for the normalized autocorrelation function becomes vanishingly small.

3. Statistics of the spectral linewidth estimator

Turning now to the experimental determination of the spectral linewidth, Γ, we investigate the best means of processing the data received from photon-counting experiments and the errors to be expected.

During the evaluation of the results presented in the previous section (see appendix) it was assumed that the intensity autocorrelation function took the form

$$g^{(2)}(\tau) = 1+C\exp(-2\Gamma\tau) \tag{27}$$

where

$$C = \frac{\sinh^2\gamma}{\gamma^2}$$

is the correction factor due to the effect of a finite sampling time T (Pike 1969, Jakeman 1970). In practice the factor C depends also on the detector area. Moreover it is altered by dead-time effects and by 'clipping'. Since the τ dependence in (27) is not changed by these corrections it is usually assumed for fitting purposes that C is an arbitrary constant independent of the linewidth. We suppose that in our experiments we have made measurements of the quantity $\hat{g}^{(2)}(\tau_i)$ for M values of the delay time, and use the method of least squares to fit the measured points to a function of the form (27); that is, we minimize the quantity

$$s = \sum_i \{\hat{g}^{(2)}(\tau_i)-1-C\exp(-2\hat{\Gamma}\tau_i)\}^2\chi_i \tag{28}$$

where $i = 1 \ldots M$, with respect to variations in C and $\hat{\Gamma}$. χ_i is the weighting which will subsequently be chosen to make the variation of $\hat{\Gamma}$ a minimum.

Performing the variations in C and $\hat{\Gamma}$, and eliminating C from the two equations so obtained, we find that $\hat{\Gamma}$ must be chosen to satisfy

$$\frac{\Sigma_i \chi_i \tau_i \exp(-2\hat{\Gamma}\tau_i)(\hat{g}_i^{(2)}(\tau_i)-1)}{\Sigma_i \chi_i (\hat{g}_i^{(2)}(\tau_i)-1)\exp(-2\hat{\Gamma}\tau_i)} = \frac{\Sigma_i \chi_i \tau_i \exp(-4\hat{\Gamma}\tau_i)}{\Sigma_i \chi_i \exp(-4\hat{\Gamma}\tau_i)}. \tag{29}$$

Expanding this relation about Γ and $g^{(2)}(\tau)$ to first order in the difference $\delta\hat{\Gamma} = \Gamma - \hat{\Gamma}$ and $\delta\hat{g} = g - \hat{g}$ leads to

$$4\langle \delta\hat{\Gamma}^2 \rangle = 4 \operatorname{Var} \hat{\Gamma}$$
$$\simeq \frac{\Sigma_i \Sigma_j \chi_i \chi_j \exp\{-2\Gamma(\tau_i+\tau_j)\}(\tau_i - \bar{\tau})(\tau_j - \bar{\tau})\langle \delta\hat{g}^{(2)}(\tau_i)\delta\hat{g}^{(2)}(\tau_j)\rangle}{\{\Sigma \chi_i \tau_i \exp(-2\Gamma\tau_i)(g^{(2)}(\tau_i)-1)(\tau_i - \bar{\tau})\}^2} \tag{30}$$

where we have defined $\bar{\tau}$ to be the right hand side of equation (29) with $\hat{\Gamma}$ replaced by Γ. In deriving (30) we have made the approximation that

$$\operatorname{Var} \hat{g}^{(2)}(\tau) = \langle [\delta\hat{g}^{(2)}(\tau_i)]^2 \rangle \ll (g^{(2)}(\tau_i) - 1)^2. \tag{31}$$

We can always satisfy the above inequality for a given τ_i by making N (or $N\gamma$) sufficiently large.

Before proceeding a digression on the correlation of the errors is in order. If we assume that the errors are uncorrelated from point to point then

$$\langle \delta\hat{g}^{(2)}(\tau_i)\delta\hat{g}^{(2)}(\tau_j)\rangle = \langle [\delta\hat{g}^{(2)}(\tau_i)]^2 \rangle \delta_{ij} \tag{32}$$

whereas if they are fully correlated

$$\langle \delta\hat{g}^{(2)}(\tau_i)\delta\hat{g}^{(2)}(\tau_j)\rangle = \{\langle [\delta\hat{g}^{(2)}(\tau_i)]^2 \rangle \langle [\delta\hat{g}^{(2)}(\tau_j)]^2 \rangle\}^{1/2}. \tag{33}$$

In what follows we assume the errors are uncorrelated. We do this because, for sufficiently small $\bar{n}$, the 'shot noise' effect of the photo-electron process dominates, and this is an uncorrelated noise process. Further, in the case $\gamma \gg 1$ the signal fluctuations are uncorrelated from sample to sample. Thus in this limit too the errors are uncorrelated. We have also performed some calculations using the weighting factors χ_i which optimize the uncorrelated case but with the assumption that the errors are fully correlated. We find that the results lie well above the errors for the uncorrelated case for $\gamma \ll 1$, but become almost equal for $\gamma \gtrsim 1$. In the fully correlated limit the normalized standard deviation in $\hat{\Gamma}$ takes the asymptotic forms

$$\frac{(\operatorname{Var} \hat{\Gamma})^{1/2}}{\Gamma} = \begin{cases} \dfrac{4 \cdot 1}{r\gamma N^{1/2}} & \bar{n}, \gamma \ll 1 \\[2ex] \dfrac{2\gamma^{1/2} e^{2\gamma}(1+r^{-1})}{(N\gamma)^{1/2}} & \gamma \gg 1 \end{cases}$$

These may be compared with results based on the uncorrelated assumption presented in the last column of table 1. By numerical computation we have in fact found that the standard deviations in the two extremes of correlation almost coincide for all $\gamma \gtrsim 1$.

Table 1. Analytic form of the error in Γ given by equation (36) in various limits

Limit	a	b	c	$\frac{(\mathrm{Var}(\hat{\Gamma}))^{1/2}}{\Gamma}$
$\gamma \ll 1$, $r\gamma \ll 1$	$\frac{r^2}{2}(1-\lg 2)$	$r^2(1-\frac{3}{4}\zeta(3))$†	$\frac{r^2}{2}\left(1-\frac{\pi^2}{12}\right)$	$\frac{4\cdot 606}{r(N\gamma)^{1/2}}$
$\gamma \gg 1$	$\frac{1-4e^{-2\gamma}+7e^{-4\gamma}}{16\gamma^3(1+1/r)^2}$	$\frac{1-4e^{-2\gamma}+10e^{-4\gamma}}{4\gamma(1+1/r)^2}$	$\frac{1-4e^{-2\gamma}+8e^{-4\gamma}}{8\gamma^2(1+1/r)^2}$	$\frac{2(\gamma)^{1/2}e^{2\gamma}(1+r)}{r(N\gamma)^{1/2}}$

† $\zeta(S)$ is the Reimann Zeta function.

Thus if we assume the errors to be uncorrelated, we will obtain a good approximation to Var $\hat{\Gamma}$ in the regions $\bar{n}, \gamma \ll 1$, and $\gamma \gtrsim 1$, whereas outside these regions we will obtain a lower bound.

Assuming (32), (30) becomes

$$\mathrm{Var}\,\hat{\Gamma} = \frac{1}{4}\frac{\Sigma_i \chi_i^2 \exp(-4\Gamma\tau_i)(\tau_i-\bar{\tau})^2\,\mathrm{Var}\,g^{(2)}(\tau_i)}{\{\Sigma_i\chi_i\tau_i(g^{(2)}(\tau_i)-1)\exp(-2\Gamma\tau_i)(\tau_i-\bar{\tau})\}^2}. \tag{34}$$

It is straightforward, if lengthy, to show that Var $\hat{\Gamma}$ is minimized if we choose

$$\chi_i = \frac{A}{\mathrm{Var}\,g^{(2)}(\tau_i)} \qquad A = \frac{1}{\sum_i \frac{\exp(-4\Gamma\tau_i)}{\mathrm{Var}\,g^{(2)}(\tau_i)}} \tag{35}$$

and that with this choice for χ, the minimum value of Var Γ is given by

$$\frac{\mathrm{Var}\,\hat{\Gamma}}{\Gamma^2} = \left\{\left(b-\frac{c^2}{a}\right)N\gamma\right\}^{-1} \tag{36}$$

where

$$b = \frac{4\Gamma^2}{N\gamma}\sum_i \frac{\tau_i^2(g^{(2)}(\tau_i)-1)^2}{\mathrm{Var}\,g^{(2)}(\tau_i)} \tag{37}$$

$$c = \frac{2\Gamma}{N\gamma}\sum_i \frac{\tau_i(g^{(2)}(\tau_i)-1)^2}{\mathrm{Var}\,g^{(2)}(\tau_i)} \tag{38}$$

$$a = \frac{1}{N\gamma}\sum_i \frac{(g^{(2)}(\tau_i)-1)^2}{\mathrm{Var}\,g^{(2)}(\tau_i)}. \tag{39}$$

We note that in the region for which these results are applicable—see the inequality (31)—the weighting is reasonably flat.

We have evaluated the expression (36) numerically for various values of γ, M and r, using the general formula for Var $g^{(2)}(\tau_i)$ given by (10), (16), (20) and (21). The results are shown as solid lines in figures 3 and 4. Figure 3 is a plot of Var $\hat{\Gamma}^{1/2}/\Gamma$ (as a percentage) against γ for values of the photon flux which range from 0·1 to 100 photons per coherence time. γ varies from 10^{-3} to 1. Figure 4 is a different way of presenting the same information for the case of an infinite number of channels. It is a plot of [Var $\hat{\Gamma}]^{1/2}/\Gamma$(as a percentage) against photon flux for various values of γ (γ varies from 10^{-2} to 1). We have taken $N\gamma = 10^4$. Since (Var $\hat{\Gamma})^{1/2}$ is proportional to $1/(N\gamma)^{1/2}$ it is a very simple matter to scale these curves for other values of $N\gamma$. It is apparent that with 20 channels, for example, the minimum error is obtained when γ lies in the range 0·06 to 0·1. We comment further on this in the discussion.

We have also evaluated analytically expression (36) for some of the limiting cases discussed in § 2 assuming that $M\gamma \gg 1$. The results are displayed in table 1. The first set of conditions shown in the table will always be satisfied for sufficiently small γ so that the value given for the error in $\hat{\Gamma}$ represents the asymptotic limit $\gamma \to 0$ in agreement with figure 3. In evaluating the large γ limit, the first two terms of the series (37), (38) and (39) must be retained because of cancellation. In this case the error increases exponentially with γ and is independent of the mean number of counts per coherence time when this is much greater than unity.

4. The effect of clipping and finite apertures

In the last section it was assumed that the constant C appearing in the model form for the intensity autocorrelation function (27) was arbitrary and independent of the linewidth. Although this is a reasonable approximation as far as the fitting procedure is concerned, the factors which determine the value of C also affect the error in $\hat{g}^{(2)}(\tau)$. In particular it is important to understand the effect of clipping and of the finite areas of the source-detector system as determined by the aperture sizes used. In general the calculations necessary are long and tedious and in view of the results presented in the previous section it seems reasonable to restrict consideration of this problem to the region $\gamma < 1$. Since in experiments so far reported clipping has been carried out in one channel at zero level we shall further restrict ourselves to this case. Certain results for an arbitrary clipping level have already been published (Jakeman *et al.* 1970a) for the case when the samples of the autocorrelation function can be regarded as independent.

In the case of finite apertures two opposing effects must be considered—the averaging out of the signal by spatial integration over the detector area which will reduce the statistical accuracy of the measurement as this area is increased, and the increase in light flux which will tend to improve the accuracy in the same circumstances. According to figure 4 the greatest reduction in error with increasing light flux occurs at small values of $\bar{n}$ where the statistics of the detection process are important and the formula (24) holds. The normalized variance of $\hat{g}^{(2)}(\tau)$ is given in this case by ($\gamma \ll 1$)

$$\frac{\operatorname{Var}\hat{g}^{(2)}(\tau)}{(g^{(2)}(\tau)-1)^2} = \frac{g^{(2)}(\tau)}{(\bar{n}f(A))^2 N} \tag{40}$$

where

$$f(A) = \frac{1}{A^2}\int_A \mathrm{d}r \int_A \mathrm{d}r' |g^{(2)}(r, t; r', t)|^2$$

has been evaluated by Scarl (1968) and for a wider range of values of A by Jakeman *et al.* (1970). Since $\bar{n} \propto A$ and $f(A)$ decreases like A^{-1} when the detector area is greater than a coherence area there is little to be gained even for small values of $\bar{n}$ by increasing the detector area beyond this value. If r is such that $\bar{n} \gtrsim 10^{-2}$, accuracy will actually be lost by exceeding a coherence area, and for large r the detector area should be reduced until $r \sim 10$ counts per coherence time.

The effect of clipping at zero is negligible in the limit where equation (40) is valid, but becomes important as $\bar{n}$ is increased. We have evaluated the variance of $\hat{G}_0^{(2)}(\tau)$ where

$$G_0^{(2)}(\tau) = \frac{1}{N} \sum_r n(t_r+\tau) n_0(t_r) \tag{41}$$

and

$$n_0(t_r) = \begin{matrix} 1 & \quad n(t_r) > 0 \\ 0 & \quad n(t_r) = 0 \end{matrix}$$

in the limit $\gamma \ll 1$. The quantity measured experimentally is normalized as discussed in § 2 and the estimator $\hat{g}_0^{(2)}(\tau)$ of the clipped intensity autocorrelation function is biased by a term proportion to $1/N$. This leads to a formula similar to equation (10):

$$\begin{aligned}\operatorname{Var}\hat{g}_0^{(2)}(\tau) = {} & \frac{\operatorname{Var}\hat{G}_0^{(2)}(\tau)}{\bar{n}^2\bar{n}_0^{\,2}} + \frac{(g_0^{(2)}(\tau))^2}{\bar{n}^2}\operatorname{Var}\hat{n} + \frac{(g_0^{(2)}(\tau))^2}{\bar{n}_0^{\,2}}\operatorname{Var}\hat{n}_0 \\ & + \frac{2(g_0^{(2)}(\tau))^2}{\bar{n}\bar{n}_0}(\langle\hat{n}\hat{n}_0\rangle - \bar{n}\bar{n}_0) - \frac{2g_0^{(2)}(\tau)}{\bar{n}^2\bar{n}_0}(\langle\hat{G}_0^{(2)}(\tau)\hat{n}\rangle - \bar{n}G_0^{(2)}(\tau)) \\ & - \frac{2g_0^{(2)}(\tau)}{\bar{n}\bar{n}_0^{\,2}}(\langle\hat{G}_0^{(2)}(\tau)\hat{n}_0\rangle - \bar{n}_0 G_0^{(2)}(\tau))\end{aligned} \tag{42}$$

where $\bar{n}_0 = \bar{n}/(1+\bar{n})$, $\hat{n}_0$ is the estimator for the clipped count rate and

$$G_0^{(2)}(\tau) = \bar{n}\bar{n}_0\left(1 + \frac{|g^{(1)}(\tau)|^2}{1+\bar{n}}\right) = \bar{n}\bar{n}_0 g_0^{(2)}(\tau) \tag{43}$$

is the single clipped intensity autocorrelation function (Jakeman and Pike 1969). In analogy with equation (5) it may be shown that

$$\operatorname{Var}\hat{G}_0(\tau) = \frac{1}{N}\operatorname{Var}(n_0(\tau)n(0)) + \frac{2}{N}\sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)\{R_0(kT) - |G_0^{(2)}(\tau)|^2\} \tag{44}$$

where

$$R_0(kT) = \langle n_0(kT+\tau)n(kT)n_0(\tau)n(0)\rangle. \tag{45}$$

This quantity may be evaluated using the following formula

$$\begin{aligned}\langle n_0(t)n(t')n_0(t_1)n(t_1')\rangle = {} & \frac{d^2}{dS'\,dS_1'}\{Q(S',S_1') - Q(S,S',S_1') - Q(S',S_1,S_1') \\ & + Q(S,S',S_1,S_1')\}_{\substack{S'=S_1'=0 \\ S=S_1=1}}.\end{aligned} \tag{46}$$

Here the generating functions of two and three variables are derived from that of the fourfold intensity probability distribution $P(I, I', I_1, I_1')$ by setting the appropriate variable equal to zero. It is not difficult to show that in the small sample time limit, the latter is given by

$$\begin{aligned}Q(S,S',S_1,S_1') = {} & \frac{1}{SS'S_1S_1'}\{qq'q_1q_1' - g_1^2q_1q' - g_2^2q'q_1' - g_3^2q_1q' - g_4^2qq_1' \\ & - g_5^2qq_1 - g_6^2qq' + g_2^2g_5^2 + g_3^2g_4^2 + g_1^2g_6^2 + 2g_1g_3g_5q_1 \\ & + 2g_1g_2g_4q_1' + 2g_2g_3g_6q' + 2g_4g_5g_6q - 2g_1g_2g_5g_6 \\ & - 2g_1g_3g_4g_6 - 2g_2g_3g_4g_5\}^{-1}\end{aligned} \tag{47}$$

where

$$q = \bar{n}+1/S \qquad q' = \bar{n}+1/S' \text{ etc}$$
$$g_1 = G^{(1)}(t-t') \qquad g_2 = G^{(1)}(t-t_1) \qquad g_3 = G^{(1)}(t-t_1')$$
$$g_4 = G^{(1)}(t'-t_1) \qquad g_5 = G^{(1)}(t'-t_1') \qquad g_6 = G^{(1)}(t_1-t_1').$$

The first term on the right hand side of (44) is evaluated in a similar way using the relation

$$\langle n_0{}^2(\tau)n^2(0)\rangle = \langle n^2\rangle - \frac{\mathrm{d}}{\mathrm{d}S}\left(\frac{\mathrm{d}}{\mathrm{d}S}-1\right) Q(S, S')\Big|_{\substack{S'=1\\S=0}} . \qquad (48)$$

The final complicated analytic form is presented in the appendix together with results for the biasing terms appearing in equation (42). Using the fitting procedure described in the last section the percentage error in the linewidth was calculated and is plotted against γ in figure 3 (broken lines). Comparison with the solid lines shows that the minimum error is shifted more rapidly towards lower values of γ as r is increased in the clipped case.

5. Discussion

The qualitative features of Var $\hat{g}^{(2)}(\tau)$ are similar to those of Var $\hat{G}^{(2)}(\tau)$ but the former is uniformly smaller by an amount which increases as r increases. Hence this discussion applies equally well to both Var $\hat{G}^{(2)}(\tau)$ and Var $\hat{g}^{(2)}(\tau)$. The main conclusion to be drawn from figure 2 and formula (21) is that Var $\hat{G}^{(2)}(\tau)$ is relatively insensitive to changes in τ. The strongest variation with delay time occurs for small values of γ when equation (18) is valid. This implies a maximum possible fall-off factor of four over the entire range of $\tau (T < \tau < \infty)$, but perhaps as little as two if r is small. For large values of γ, Var $\hat{G}^{(2)}(\tau)$ is essentially independent of τ. As far as the experimenter is concerned it is a reasonable approximation to take a constant weighting when carrying out the type of fitting described in the last section.

Excluding values of $\gamma \gtrsim 1$ which give rise to large errors, little is to be gained by fitting the autocorrelation function beyond two or three coherence times. In fact the results for M channels show that the minimum error is obtained when T is chosen so that $M\Gamma T \sim$ 2–3 when r lies in the range 0·1 to 1, and $M\Gamma T \sim$ 1–2 when r lies in the range 10 to 100. The minimum is much sharper when M is small than when M is large. Moreover, if the maximum delay time and the total experiment time are regarded as fixed quantities it is evident from figure 3 that the error in linewidth cannot be reduced significantly by improving the resolution of the instrument beyond a certain value of T which depends on the number of counts per coherence time. In this region, roughly defined by $\bar{n} < 10^{-2}$,the increase in the number of points of the autocorrelation function used for fitting merely compensates for the increased error in these points for small γ depicted in figure 2. Inspection of figure 4 on the other hand indicates that accuracy cannot be improved by increasing indefinitely the number of counts per coherence time; saturation takes place for $r \gtrsim 10$.

The above results have implications in connection with the optimum detector area to be used as mentioned in the last section. The number of counts per coherence time can be increased by increasing the detector area, but at the same time information is lost due to spatial integration over the receiver surface. Evidently if $r \sim 10$ there will be a net increase in error if the detector area is increased and if $r > 10$ it is advantageous to reduce the detector area until r is of this order. However, when the photon statistics dominate the accuracy of the measurement, it may be advantageous to increase the detector area. Signal to noise ratios have been evaluated for large detector areas (i.e. $A \gg$ coherence area and consequently Gaussian intensity statistics) by Haus (1967), Benedek (1968) and Cummins and Swinney (1969). Since $\bar{n}$ is proportional to A, however, the error introduced by (40) cannot be reduced significantly by increasing the detector area in this limit. For small values of A the function $f(A)$ decreases rather more slowly and accuracy may be improved by increasing A up to about a coherence area, where the effect saturates, or to a value of A for which the $1/\bar{n}$ dependence present in equation (40) ceases to dominate the error.

The effect of clipping on the results is shown by the broken curves in figure 3. Strictly speaking these are only valid in the limit $\gamma \ll 1$ but in practice represent a good approximation for values of $\gamma \lesssim 0{\cdot}3$. For small values of r, which give rise to small values of $\bar{n}$ over the range of γ values plotted, clipping at zero has little effect on the results as expected. A considerable increase in the percentage error in line-width is caused by clipping at zero when $\bar{n}$ is large, however, due to the loss of information in these circumstances. One unexpected feature of the curves for $r = 100$ is that the clipped results actually fall below the unclipped ones for a range of small values of γ. The origin of this effect is revealed by comparing the errors in the clipped and unclipped autocorrelation functions assuming that the samples are uncorrelated. In this particularly simple case the normalized error in the unclipped autocorrelation function is given from equation (16) in the limit $\gamma \ll 1$ by

$$\frac{\mathrm{Var}\,\hat{G}^{(2)}(\tau)}{(G^{(2)}(\tau)-\bar{n}^2)^2} = \left(3+\frac{4}{\bar{n}}+\frac{1}{\bar{n}^2}\right)|g^{(1)}(\tau)|^{-4}+\left(14+\frac{8}{\bar{n}}+\frac{1}{\bar{n}}\right)|g^{(2)}(\tau)|^{-2}+3 \tag{49}$$

whereas when clipping is carried out at zero in one channel we have (Jakeman *et al.* 1970 equation (4))

$$\begin{aligned}\frac{\mathrm{Var}\,G_0{}^{(2)}(\tau)}{(G_0{}^{(2)}(\tau)-\bar{n}\bar{n}_0)^2} &= \left(\frac{1}{\bar{n}^2}+\frac{5}{\bar{n}}+5\bar{n}+\bar{n}^2\right)|g^{(1)}(\tau)|^{-4}\\ &\quad+\left(\frac{1}{\bar{n}^2}+\frac{6}{\bar{n}}+7+2\bar{n}\right)|g^{(1)}(\tau)|^{-2}-(3+2\bar{n}).\end{aligned} \tag{50}$$

When $\bar{n} \ll 1$, (49) and (50) become identical, whilst for sufficiently large $\bar{n}$ (50) is greater than (49) as is also the case when τ is large enough for the terms in $|g^{(1)}(\tau)|^{-4}$ to dominate. However, for $\bar{n} \sim 1$ and sufficiently small τ, (50) can actually be less than (49). This is not entirely surprising since we are comparing the variances of two different quantities and the contribution to the variance arising from the correlation of samples shows a similar behaviour. Owing to the kind of fitting procedure we have used, however, the error in linewidth depends through (36)–(39) only on these normalized variances so that clipping can improve the accuracy of a linewidth measurement in certain circumstances. We have then lost more information in the unclipped case by assuming C in equation (22) to be an arbitrary constant than we have lost in the clipped case due both to this assumption and to the clipping process itself.

The effect of clipping in one or both channels at values different from zero is a problem of some interest which has still to be solved for the sampling scheme shown in figure 1. Although some preliminary results for the case when the samples can be regarded as uncorrelated have already been presented (Jakeman *et al.* 1970b), analytical solution of the problem is in general difficult and computer simulation techniques, commonly applied to analogous problems in the radar field seem more appropriate. Such calculations are under way and we hope to be able to use them to confirm the accuracy of the rather lengthy calculations presented here, by comparing a few particular cases, as well as to extend the results to higher clipping levels. We have regarded computer simulation as a last resort for, although not difficult to carry out, the number of parameters in the problem and the large numbers of samples to be calculated lead to lengthy computation times for each error value and an overall picture is not easily obtained.

Appendix

Here we shall set down in full various formulae which we have derived and used during the course of our calculations but which were too lengthy to include in the main text.

(i) Var $\hat{G}^{(2)}(\tau)$. In order to evaluate equation (5) (§ 2) we need to know $R(kT_p)$ which is given by (12) if $kT_p = \tau$ for some integer value of k. Using the factorization properties of Gaussian light

$$\frac{\langle I(t_1)I(t_2)I(t_3)I(t_4)\rangle}{\langle I\rangle^4} = 1+g_{12}^2+g_{13}^2+g_{14}^2+g_{23}^2+g_{24}^2+g_{34}^2+g_{12}^2g_{34}^2+g_{13}^2g_{24}^2 +g_{14}^2g_{23}^2+2(g_{23}g_{34}g_{24}+g_{12}g_{23}g_{13}+g_{14}g_{12}g_{24}+g_{13}g_{34}g_{24} +g_{12}g_{23}g_{34}g_{14}+g_{12}g_{13}g_{24}g_{34}+g_{12}g_{14}g_{24}g_{23}) \quad \text{(A1)}$$

and

$$\frac{\langle I(t_1)I(t_2)I(t_3)\rangle}{\langle I\rangle^3} = 1+g_{12}^2+g_{23}^2+g_{31}^2+2g_{12}g_{23}g_{13} \quad \text{(A2)}$$

where $g_{12} = g(t_1-t_2) = \exp(-\Gamma|t_1-t_2|)$ for a Lorentzian spectrum. The time integrals in the expression for $R(kT_p)$ may be performed using (A1) and (A2) and the sums appearing on the right hand side may be performed to give

$$\frac{1}{\bar{n}^4}\sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)\{R(kT_p)-|G^{(2)}(\tau)|^2\} = F\left[\frac{(3Y+1)(3X+1)}{(1-Y)}+4(m-2)X-1 +F\left\{\frac{3X^2+Y^2}{(1-Y^2)}+3(m-2)X^2\right\}\right] - \frac{F}{N}\left[\frac{Y\{2(1+7X)-Y^N(1+3X)^2/X\}}{(1-Y)^2} +m\frac{(Y+4X+1)}{(1-Y)}+2m(m-3)X-m+F\left\{\frac{Y^2(1-4Y^{2N}+3X^2)}{(1-Y^2)^2}+\frac{3mX^2}{(1-Y^2)} +\frac{3m(m-3)X^2}{2}\right\}\right] + \frac{1}{\gamma}(1-m/N)\left\{\left(1-\frac{1-Z}{2\gamma}\right)(1+3FX^2)+4X\left(F-\frac{1-Z}{2\gamma}\right)\right\} +\frac{1}{\bar{n}}\left(1-\frac{m}{N}\right)\{1+FX(2+3X)\} \quad \text{(A3)}$$

where $Y = \exp(-2\Gamma T_p)$, $X = \exp(-2\Gamma\tau)$, $F = (\sinh^2\gamma)/\gamma^2$, $Z = \exp(-2\gamma)$ and we have assumed that $mT_p = \tau$ for some integer m. If $T_p > \tau$ we obtain

$$\frac{1}{\bar{n}^4}\sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)\{R(kT_p)-|G^{(2)}(\tau)|^2\} = F\left(\frac{Y}{1-Y}\right)\left[\frac{(1+3X)^2}{X}+\frac{4FY}{(1+Y)} -\frac{1}{N}\frac{(1-Y^N)}{1-Y}\left\{\frac{(1+3X)^2}{X}+\frac{4FY(1+Y^N)}{(1+Y)^2}\right\}\right] \quad \text{(A4)}$$

(A4) is given for completeness although it is not used in the present work.

(ii) Bias terms: Var $g(\tau)$. We quote only the results

$$\operatorname{Var}\hat{n} = \frac{\bar{n}^2}{N}\left[\left(\frac{1}{\bar{n}}+\frac{1}{\gamma}\right) - \frac{\{1-\exp(-2\gamma N)\}}{2N\gamma^2}\right] \tag{A5}$$

$$\begin{aligned}\langle \hat{G}^{(2)}(\tau)\hat{n}\rangle - G^{(2)}(\tau)\bar{n} &= \frac{2\bar{n}^3}{N}\left\{\frac{g^{(2)}(\tau)}{\bar{n}} + FX\left(m-1+\frac{2}{\gamma}-\frac{2Z}{1-Z}\right)\right\} \\ &+ \frac{\bar{n}^3}{N^2}\left[m - FX\left\{\frac{2Z}{1-Z}\left(m\left(1+\frac{1}{X}\right)+1\right) - 2 + (m+1)(m-2)\theta(m-1)\right\}\right. \\ &\left. - \frac{1}{\gamma^2}\left\{\tfrac{1}{2} + X\left(1+\frac{Z}{2}\right) - \frac{Z^N}{4}\frac{(1+X)(1+3X)}{X}\right\}\right]\end{aligned} \tag{A6}$$

where $\theta(m-1) = \begin{matrix} 1 & m > 1 \\ 0 & m \leqslant 1. \end{matrix}$

(iii) Var $\hat{G}_0^{(2)}(\tau)$. This quantity is calculated as indicated in § 4 retaining only the terms of order γ^{-1} in the sum appearing in equation (44) and neglecting all terms of order N^{-2}. The final result may be written in the form

$$\operatorname{Var}\hat{G}_0(\tau) = \frac{1}{N}\operatorname{Var}(n_0(\tau)n(0)) + \frac{1}{N\gamma}F(\bar{n},\tau) + \frac{2}{N}\langle n_0(2\tau)n(\tau)n_0(\tau)n(0)\rangle \tag{A7}$$

where

$$\operatorname{Var}(n_0(\tau)n(0)) = \bar{n}_0^2(1+3\bar{n}+\bar{n}^2) + \bar{n}_0^3X(\bar{n}^{-1}+5+2\bar{n}) - n_0^4X^2(3+2\bar{n}) \tag{A8}$$

$$\langle n_0(2\tau)n(\tau)n_0(\tau)n(0)\rangle = \bar{n}_0^3\{(1+\bar{n})^2 + (1+\bar{n})(2+\bar{n})X + 3(1+\bar{n})X^2 - 2\bar{n}X^3\} \tag{A9}$$

$$\begin{aligned}F(\bar{n},\tau) &= 2\Gamma\tau\frac{\bar{n}_0^6}{\bar{n}^2}X\{2(1+\bar{n})(\bar{n}^2+2\bar{n}+2) + X(\bar{n}^2-2\bar{n}+3)\} \\ &+ \left[1 + \frac{\bar{n}_0^6}{\bar{n}^3}X\{\bar{n}X(\bar{n}^2-2\bar{n}+3) - 2(1+\bar{n})(1+\bar{n}+\bar{n}^2)\}\right]\lg(1-\bar{n}_0^2X) \\ &- \left[1+\bar{n}_0^2+2\frac{\bar{n}_0^6}{\bar{n}^3}X\{\bar{n}X(2+\bar{n}^2) - (1+\bar{n})(2+3\bar{n}+2\bar{n}^2)\}\right]\lg(1-\bar{n}_0^2) \\ &+ 2\frac{\bar{n}_0^2}{\bar{n}^3}(1-\bar{n}_0^2X)^{-1} + \frac{\bar{n}_0^3}{\bar{n}^4}(\bar{n}^6+3\bar{n}^5-\bar{n}^4-\bar{n}^3-2\bar{n}-2) \\ &+ \frac{\bar{n}_0^6}{\bar{n}^5}X\frac{(18\bar{n}^4+17\bar{n}^3+4\bar{n}^2-3\bar{n}-2)}{(1-\bar{n}_0^2)} - 2\frac{\bar{n}_0^8}{\bar{n}^5}X^2\frac{(6\bar{n}^3+2\bar{n}^2+\bar{n}+1)}{(1-\bar{n}_0^2)^2}.\end{aligned}$$

$X = \exp(-2\Gamma\tau)$, $\bar{n}_0 = \bar{n}/(1+\bar{n})$ as before. There is some inconsistency in neglecting the terms independent of γ in evaluating the sum in equation (44) whilst retaining such terms in (A8) and (A9) (those terms proportional to $\bar{n}^4$). For sufficiently small γ, however, this is not important. Calculations in fact show that the 'small γ' approximation leading to the above results is valid for values of γ less than about 0·3, and thus in the region of chief interest (figure 3).

(iv) Bias terms: Var $\hat{g}_0^{(2)}(\tau)$.

We quote the results in the 'small γ' limit, neglecting terms of order N^{-2}

$$N\,\mathrm{Var}\,\hat{n}_0 = \frac{\bar{n}_0^2}{\bar{n}} - \left(\frac{\bar{n}_0^2}{\bar{n}^2\gamma}\right) \lg(1-\bar{n}_0^2)$$

$$N(\langle \hat{n}\,\hat{n}_0\rangle - \bar{n}\,\bar{n}_0) = \bar{n}_0^2\,(1+1/\gamma)$$

$$N(\langle \hat{G}_0^{(2)}(\tau)\hat{n}\rangle - G_0^{(2)}(\tau)\bar{n}) = \bar{n}_0^2\left(2+\bar{n}+\frac{2\bar{n}_0X}{\bar{n}}\right)+\frac{\bar{n}_0^3}{\gamma}[(1+\bar{n})(2+\bar{n})+2X \times\{1+\Gamma\tau(1+\bar{n})\}]$$

$$N(\langle \hat{G}_0^{(2)}(\tau)\hat{n}_0\rangle - G_0^{(2)}(\tau)\bar{n}_0) = \frac{\bar{n}_0^2(\bar{n}+2)}{\bar{n}} - \bar{n}_0^2X - \frac{\bar{n}_0^3}{\bar{n}^2}\frac{(1-\bar{n}_0X)}{(1-\bar{n}_0^2X)^2} - \frac{n_0^4}{\bar{n}^3} \times \frac{(1+2\bar{n}-2\bar{n}X)}{(1-\bar{n}_0^2)^2} + \frac{1}{\gamma}\left[\frac{2\Gamma\tau X\,\bar{n}_0^5}{\bar{n}^2} + \left(\frac{\bar{n}_0^2}{\bar{n}}\right) \times\left\{\frac{\bar{n}_0^3(1+\bar{n}+\bar{n}^2)X}{\bar{n}^2}-1\right\}\lg(1-\bar{n}_0^2)+\left(\frac{\bar{n}_0^5}{\bar{n}^2}\right) \times X\lg(1-\bar{n}_0^2X)+\bar{n}_0^3\left\{1+\frac{2X}{1+2\bar{n}}\right\}\right].$$

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